# Probability

Finite Math

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(c) Suppose we only want to know if the coins match or not. Then we can use
$$S_3 = \{M, D\}$$
as a sample space where $M$ stands for match and $D$ stands for do not match.

# Now You Try It!

## Example

*An experiment consists of recording the boy-girl composition of a three-child family. What would be an appropriate sample space if:*

(a) *we are interested in the genders of the children in the order of their births? (A tree diagram can help.)*

(b) *we are interested only in the number of girls in family?*

(c) *we are interested only in which gender there are more of?*

(d) *we are interested in all three items from (a)-(c)?*

# Now You Try It!

## Solution

(a)

$$S_1 = \{GGG, GGB, GBG, BGG, GBB, BGB, BBG, BBB\}$$

(b)

$$S_2 = \{0, 1, 2, 3\}$$

(c)

$$S_3 = \{G, B\}$$

(d) *Use* $S_1$.

# Dice Rolling Outcomes

## Example

*Suppose we run an experiment by rolling two dice. What is the most fundamental sample space for this experiment? Give the event for each of the following outcomes. Which are simple events?*

(a) *A sum of* 7 *turns up.*
(b) *A sum of* 11 *turns up.*
(c) *A sum less than* 4 *turns up.*
(d) *A sum of* 12 *turns up.*
(e) *A sum of* 5 *turns up.*
(f) *A sum which is prime turns up.*
(g) *A sum of* 2 *turns up.*

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## Definition (Probabilities of Simple Events)

*Given a sample space*

$$S = \{e_1, e_2, ..., e_n\}$$

*with n simple events, to each simple event $e_i$ we assign a real number, denoted by $P(e_i)$, called the probability of event $e_i$. These numbers can be assigned in an arbitrary manner as long as the following two conditions are satisfied:*

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*Any probability assignment that satisfies Conditions 1 and 2 is said to be an acceptable probability assignment.*

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Notice that this satisfies the two conditions of the previous definition. Is this a reasonable assignment of probabilities? Ostensibly yes since there are only two outcomes of a coin flip and there is no reason to doubt that the coin is *fair*, i.e., the two outcomes are equally likely.

# Alternative Examples

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Another assignment we could technically make is

$$P(H) = 1 \qquad P(T) = 0.$$

While this does fit the rules of an acceptable probability assignment, it is not *reasonable* in this case, unless the coin had two heads.

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since $P(H) + P(T) = 1.4 > 1$.

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(c) *If $E$ is a compound event, then $P(E)$ is the sum of the probabilities of all the simple events in $E$.*

(d) *If $E$ is the sample space $S$, then $P(E) = P(S) = 1$ (this is a special case of part (c).)*

# Example

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*Refer back to the example where we roll two dice. If we assume that every simple event is equally likely:*

(a) *What is the probability of a simple event happening?*

(b) *What are the possible numbers that the two dice could add up to?*

(c) *What are the probability of each of the events in part (b) happening?*

# Equally Likely Assumption

When we were talking about assigning probabilities of 0.5 to heads and 0.5 to tails for flipping a coin, and a probability of $\frac{1}{6}$ for any number to come up when rolling a 6-sided die, we are making an assumption on the probabilities of the experiment called an *equally likely assumption*.

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Generally, if the sample space is

$$S = \{e_1, e_2, ..., e_n\},$$

we assign to each $e_i$ a probability of $\frac{1}{n}$ since there are $n$ possible outcomes and we want each of them to be equally likely. This gives us the following theorem...

# Equally Likely Assumption

**Theorem (Probability of an Arbitrary Event under an Equally Likely Assumption)**

*If we assume that each simple event in a sample space $S$ is equally likely to occur, then the probability of an arbitrary event $E$ in $S$ is given by*

$$P(E) = \frac{n(E)}{n(S)},$$

*the number of elements in $E$ divided by the number of elements in $S$.*

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We saw this theorem in action when we found the theoretical probabilities for rolling a number on a pair of dice.

# Empirical Approach

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$$P(H) = \frac{373}{1000} \qquad P(T) = \frac{623}{1000}$$

since it reflects the results of an extensive experiment.

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